Optimism in Games with Non-Probabilistic Uncertainty

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***Abstract*— The paper studies one-shot two-player games with non-Bayesian uncertainty. The players have an *attitude* that ranges from optimism to pessimism in the face of uncertainty. Given the attitudes, each player forms a belief about the set of possible strategies of the other player. If these beliefs are consistent, one says that they form an *uncertainty equilibrium*. One then considers a two-phase game where the players first choose their attitude and then play the resulting game. The paper illustrates these notions with a number of games where the approach provides a new insight into the plausible strategies of the players.**

I. INTRODUCTION

We study a one-shot non-cooperative game of two rational players with non-probabilistic information uncertainty. Specifically, we assume that the set of possible values of the uncertain parameter is known, but that no prior distribution is available. Thus, instead of the more traditional Bayesian approach where user maximize their expected reward, here, players have an *attitude* that models their risk-aversion. An optimistic (respectively, pessimistic) player assumes that the other player will choose a strategy that is beneficial (respectively, detrimental) to her. A moderately optimistic player makes an intermediate assumption. However, in contrast with other approaches, we assume that the players choose their attitude by analyzing the consequences of their choice, instead of assuming that their risk-aversion is predetermined.

Many researchers have explored non-Bayesian models of uncertainty. Knight [7] raised questions about the suitability of probabilistic characterizations of uncertainty in some situations. Allais' parodox and Ellsberg's paradox [5] are examples of situations where decision makers violate the expected utility hypothesis. More recently, Binmore [2] and Lec and Leroux [8] explored more philosophical questions on inaccuracy, arbitrariness, and illegitimacy of Bayesianism in games. The behavioral sociology literature also reports that Bayesian strategies fail to occur in some real world games [10]. A few noteworthy experiments demonstrate a *certainty effect* where people prefer less uncertain events, a *refection effect* where people respond differently to gain and loss [1], and *preference reversals* where people show different valuations when they buy and when they sell the same lottery [3]. See also [9] for a related discussion of the modeling of uncertainty through a family of probability distributions.

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Different players may have a different objective in the face of uncertainty. Some popular choices include minimax regret, maximin pessimism or maximax optimism. Instead of a fixing a player's optimization objective, we allow a rational player to choose somewhere between worst case and best case. We parametrize a player's subjective decision criterion as a convex combination of pessimism and optimism with parameter π, and we call it a player's *attitude* against uncertainty. Hurwicz (1951) [6] proposed a similar convex combination criterion for a single agent decision making problem. However, one crucial aspect of this study is that the attitude is not fixed ahead of time. Instead, the players choose their attitude *strategically*. Thus, arbitrariness in choosing a subjective decision criterion disappears while flexibility is maintained. For instance, the players may realize that the only rational attitude is to be optimistic because it is the only Nash equilibrium in a two-stage game where the first stage is to choose the attitude. More generally, there may be a set of attitudes for each player from which it is not rational to deviate unilaterally. In such a case, the model provides some information about how to behave rationally in the face of uncertainty.

Section II develops a model of two non-cooperative players with non-probabilistic parameter uncertainty, and introduces the notions of attitude and uncertainty equilibrium. Section III presents examples for which the approach provides a new insight into the strategies. Section IV proves the existence condition of an uncertainty equilibrium and relates it to a Nash equilibrium of the corresponding full information game. Section V proves that at least one player should not be pessimistic. Section VI concludes the paper.

II. UNCERTAINTY EQUILIBRIUM

The section defines the model of game with uncertainty. It then introduces the notion of uncertainty equilibrium for players that have specific attitudes. The section then defines the two-phase game. First, we define a reference game with full information.

Definition 1 (Certainty Game $\mathcal{G}_o$)
Two non-cooperative, selfish and rational players $i = 1, 2$ and $j = 3 - i$ play a game with strategies $x := (x_1, x_2) \in X_{1,o} \times X_{2,o}$, where $X_{i,o} \subset \mathbb{R}$ is i's closed bounded strategy interval. Player i has type $\theta_i \in \mathbb{R}$. The reward of player i is real-valued $u_i(x, \theta_i)$. This is a full information game with common knowledge about u_i, $X_{i,o}$, and θ_i for all i. We assume that this game is such that $u_i(x, \theta_i)$ is continuous in (x, θ_i), has a unique maximizer $x_i(x_j, \theta_i)$ for every (x_j, θ_i), and has at least one pure Nash equilibrium.

We now consider the game with uncertainty about the opponent's type.

Definition 2 (Uncertainty Game $\mathcal{G}$)
Player i knows her own true type θ_i but only that $\theta_j \in \Theta_j$ for $j = 3 - i$, where Θ_j is a closed bounded real interval, and this is common knowledge. To avoid triviality, Θ_j is assumed to be of non-zero length unless specified otherwise.

The goal of the paper is to study the notion of equilibrium in such a situation. Our approach is non-Bayesian. That is, we do assume neither a known posterior distribution of the parameters nor the existence of a common prior distribution.

We start with a simple approach to refine the set of rational strategies. Assume that it is known that player i chooses $x_i \in X_i$. It may be reasonable to believe that player j will choose a strategy $x_j(x_i, \theta_j)$ for some $x_i \in X_i$. Since player i does not know θ_j, she may then believe that player j chooses $x_j \in \phi_j(X_i)$ where

$$\phi_j(X_i) := \{x_j(x_i, \theta_j) \mid x_i \in X_i, \theta_j \in \Theta_j\}. \tag{1}$$

These considerations lead to the following definition.

Definition 3 *The sets X_1, X_2 are* consistent *if $X_j = \phi_j(X_i)$ for $i = 1, 2$ and $j = 3 - i$.*

The consistent sets form a product space of strategies beyond which no rational player plays. Although the sets X_i are smaller than the original strategy spaces $X_{i,o}$, they may be large and provide little recommendation on the strategies the players should choose. Moreover, one may question whether the players will choose strategies in the consistent sets.

A. Optimism and Pessimism

We now develop a different formulation of the game that considers the *attitudes* $\pi = (\pi_1, \pi_2) \in [0, 1]^2$ of players in the face of uncertainty.

Definition 4 (Game with Attitudes π: $\mathcal{G}(\pi)$)
If it is known that player j chooses $x_j \in X_j$, then player i chooses $x_i \in X_{i,o}$ to maximize

$$f_i(x_i, X_j, \theta_i, 1) := \max_{x_j \in X_j} u_i(x, \theta_i)$$

if she is optimistic and to maximize

$$f_i(x_i, X_j, \theta_i, 0) := \min_{x_j \in X_j} u_i(x, \theta_i)$$

if she is pessimistic. In general, for $0 \le \pi_i \le 1$, if player i has attitude π_i, she chooses $x_i \in X_{i,o}$ to maximize

$$\begin{aligned} &f_i(x_i, X_j, \theta_i, \pi_i) \\ &:= \pi_i \max_{x_j \in X_j} u_i(x, \theta_i) + (1 - \pi_i) \min_{x_j \in X_j} u_i(x, \theta_i). \end{aligned} \tag{2}$$

We primarily study a discrete attitude space $\pi_i \in \{0, 1\}$, and later use the continuous attitude space $\pi_i \in [0, 1]$ in developing the notion of robust attitude.

Designate by $r_i(X_j, \theta_i; \pi_i)$ the set of maximizers of $f_i(x_i, X_j, \theta_i, \pi_i)$.

$$r_i(X_j, \theta_i, \pi_i) := \arg \max_{x_i \in X_{i,o}} f_i(x_i, X_j, \theta_i, \pi_i). \tag{3}$$

Since player j does not know θ_i, she assumes that $x_i \in \psi_i(X_j; \pi_i)$ where

$$\psi_i(X_j; \pi_i) := \bigcup_{\theta_i \in \Theta_i} r_i(X_j, \theta_i; \pi_i). \tag{4}$$

B. Uncertainty Equilibrium

We then have the following definition.

Definition 5 (Uncertainty Equilibrium of $\mathcal{G}(\pi)$)
The pair of sets (X_1, X_2) is an uncertainty equilibrium for players with attitudes π, *if $X_i = \psi_i(X_j; \pi_i)$ for $i = 1, 2$ and $j = 3 - i$.*

Moreover, if the uncertainty equilibrium is unique, we consider that player i plays $x_i \in r_i(X_j, \theta_i; \pi_i)$ to maximize her *interim anticipated reward* $f_i(x_i, X_j, \theta_i, \pi_i)$. If the corresponding x_i is unique and equal to $x_i(\theta_i, \pi)$, it results in *actual (ex-post) rewards* $U_i := u_i(x_i(\theta_i, \pi), x_j(\theta_j, \pi), \theta_i)$. If the context is clear, we simplify as $U_i(\pi) := u_i(x_i(\pi), x_j(\pi), \theta_i)$ where $x_i(\pi) = x_i(\theta_i, \pi)$.

C. Attitude Game

Is it preferable to be optimistic or pessimistic? To answer this question, we consider a two-stage game.

Definition 6 (Attitude Game $\mathcal{A}$)
In the first stage, the players choose their attitudes $(\pi_1, \pi_2) \in \{0, 1\}^2$. In the second stage, they play $\mathcal{G}(\pi)$ and get the rewards $U_i(\pi)$.

If $\pi = (0, 0)$ is a unique Nash equilibrium for the two-stage game, we conclude that the players should be pessimistic Moreover, the analysis then specifies precisely how they should choose their second stage strategy. The situation is similar if any $\pi \in \{0, 1\}^2$ is a unique Nash equilibrium attitude. A player i's attitude π_i^* is said to be *dominant* if for any π_j and θ_j, $j = 3 - i$,

$$U_i(\pi_i^*, \pi_j) \ge U_i(\pi_i, \pi_j)$$

for all π_i.

In contrast with traditional approaches, we do not consider that players have a fixed attitude (as a *type*). Instead, they decide whether to be optimistic or not given the game. They choose their attitudes by analyzing the game instead of being driven by a preordained risk aversion.

As we show in the following sections, there are games where this approach enables to rationalize specific strategies under uncertainty.

III. Examples

The first example is a game with negative externality. In this game, the players should be optimistic even when they are uncertain about the opponent's type. The second example is a Cournot duopoly game [4] with uncertainty. For this game, we study conditions for the existence of dominant attitudes, and robust attitudes. For clarity, the algebraic derivations are in the appendix.

A. A Game with Negative Externality

Consider two players $i = 1, 2$ consume resource $x_i \in [0, 1]$ to gain benefit but also the consumption degrades the quality of the environment which affects both players. The player's reward is defined to be the benefit minus the degradation of the environment quality. The benefit is assumed to be proportional to the consumption. The environment degrades exponentially in sum of players' consumption ($\exp\{x_1 + x_2\}$), via scaling factor $\exp\{-\theta_i\}$, where θ_i^{-1} captures i's susceptibility to the environmental degradation. θ_i is private information. $x_i \in [0, 1], \theta_i \in [\alpha, \beta]$ for some $0 < \alpha < 2\alpha < \beta < 1$. Operator i's reward is

$$u_i(x, \theta_i) = x_i - \exp\{-\theta_i + x_i + x_j\}.$$

(One may add a constant to make the rewards positive.)

Theorem 1 *Players should be optimistic and choose the consumption levels $x_i = \theta_i - \alpha/2$ for $i = 1, 2$. In contrast, if θ_1, θ_2 are fully known and $\theta_1 < \theta_2$, then the only Nash strategy is $(x_1, x_2) = (0, \theta_2)$.*

For this game, the only consistent sets (see Definition 3) are $X_1 = X_2 = [0, \beta]$, which provides little information about the strategies of the players.

B. Cournot Duopoly Game

1) Full Information Case: For $i = 1, 2$, selfish and rational player i produces a non-negative quantity x_i of homogeneous items with a non-negative production cost $\theta_i \in [0, 1/2]$ per item. The selling price per item is $(1 - x_1 - x_2)^+$ where $y^+ = \max\{y, 0\}$ for $y \in \Re$. Accordingly, the reward (profit) of player i is $u_i(x, \theta_i)$ defined as follows:

$$u_i(x, \theta_i) := x_i(1 - x_1 - x_2)^+ - \theta_i x_i \tag{5}$$

where $x = (x_1, x_2)$.

Player i's strategy is the quantity x_i to produce. The value of x_i that maximizes $u_i(x, \theta_i)$ is $x_i = (1 - \theta_i - x_j)/2$, for $i = 1, 2$ and $j = 3 - i$. The unique solution of these equations is the Nash equilibrium $x^\star := (x_1^\star, x_2^\star)$ where

$$x_i^\star = (1 - 2\theta_i + \theta_j)/3. \tag{6}$$

The corresponding utilities are

$$u_i^\star = x_i^{\star 2}. \tag{7}$$

Note that the pair $x = (x_1, x_2)$ that maximizes $u_{social} := \sum_{i=1,2} u_i(x, \theta_i)$ is $((1 - \theta_1)/2, 0)$ when $\theta_1 < \theta_2$. This "social optimum" is quite different from the Nash equilibrium. There

$$u_{social} = (1 - \theta_1)^2/4. \tag{8}$$

2) Bayesian Uncertainty Case: In a Bayesian model, one assumes that θ_1 and θ_2 are independent with known distributions; each player $i = 1, 2$ knows θ_i and only the distribution of θ_j for $j = 3 - i$, and this is common knowledge. In that case,

$$E[u_1(x, \theta_1)|x_1, \theta_1] = x_1(1 - x_1 - E[x_2|x_1, \theta_1]) - \theta_1 x_1$$

and this expression is maximized by

$$x_1 = (1 - E[x_2|x_1, \theta_1] - \theta_1)/2 = (1 - E(x_2) - \theta_1)/2.$$

The last expression follows from the observation that x_2 is only a function of θ_2 which is independent of θ_1. Consequently, for $i = 1, 2$,

$$E(x_i) = (1 - E(x_j) - \mu_i)/2 \text{ where } \mu_i := E(\theta_i).$$

Solving this system of two equations, we find

$$E(x_1) = (1 - 2\mu_1 + \mu_2)/3 \text{ and } E(x_2) = (1 - 2\mu_2 + \mu_1)/3.$$

Accordingly, for $i = 1, 2$,

$$x_i = (2 - 3\theta_i - \mu_i + 2\mu_j)/6 \text{ where } j = 3 - i. \tag{9}$$

This solution is a unique Bayesian Nash equilibrium. Note that player i's strategy maximizes her *interim expected utility* $E[u_i(x, \theta_i)|x_i, \theta_i]$, rather than the *ex post utility* $u_i(x, \theta_i)$, which i cannot compute.

3) Game with Attitudes: One assumes that, for $i = 1, 2$, player i knows θ_i but only that $\theta_j \in \Theta_j := [\alpha_j, \beta_j]$ for $j = 3 - i$ where $\beta_j \leq 1/2$. This is common knowledge. Moreover, player i has attitude $\pi_i \in [0, 1]$. The following result is shown in the appendix.

Theorem 2 *The unique uncertainty equilibrium with attitudes π is the pair of intervals $\mathcal{B}[s_i, t_i] := [s_i - t_i/2, s_i + t_i/2]$ for $i = 1, 2$, where*

$$s_i = \frac{1}{3}\Delta_j \pi_i - \frac{1}{6}\Delta_i \pi_j + \frac{1}{12}(4 - 3\beta_i - 5\alpha_i + 4\alpha_j) \tag{10}$$

and $\Delta_i := \beta_i - \alpha_i$ and $t_i = (\beta_i - \alpha_i)/4$. The strategies that maximize the interim anticipated rewards are

$$x_i^*(\pi) = \frac{1}{3}\Delta_j \pi_i - \frac{1}{6}\Delta_i \pi_j + \lambda_i, \tag{11}$$

where $\lambda_i = (2 - \alpha_i + 2\alpha_j - 3\theta_i)/6$.

Based on this analysis, one considers the two-stage game $\mathcal{A}$. The following lemma is proved in the appendix.

Lemma 1 (Dominant attitude)
Let $\underline{\theta}_i := \frac{1}{3}(2 - \beta_i + 4\alpha_j - 2\beta_j)$ and $\overline{\theta}_i := \frac{1}{3}(2 - \alpha_i + 4\beta_j - 2\alpha_j)$. Assume that the attitude space is discrete $\Pi = \{0, 1\}$.

1) *If $\theta_i \leq \underline{\theta}_i$, then optimism is a dominant strategy for player i.*
2) *If $\theta_i \geq \overline{\theta}_i$, then pessimism is a dominant strategy for player i.*
3) *If $\underline{\theta_i} < \theta_i < \overline{\theta}_i$, then there is no dominant strategy for player i.*

In particular, if $\beta_i < 1/3$ for $i = 1, 2$ (i.e., if the unit production costs are sufficiently low), both players should be optimistic.

The game is said to be *symmetric* if $u_1 = u_2$ and $\Theta_1 = \Theta_2$. The following result corresponds to a symmetric game.

Theorem 3 *Consider the game $\mathcal{A}$ with $\Theta_1 = \Theta_2 = [\alpha, \beta]$ where $\beta > \alpha$.*

1) (PP) is never a Nash equilibrium.
2) (PP) is pareto efficient.
3) (PP) is pareto superior to (OO).
4) O is the dominant strategy if $\beta \le \max(1/3, 2\alpha)$. Then (OO) is the only Nash equilibrium.

Together with 1), 2), and 3), the condition in the last part makes the attitude game a Prisoner's Dilemma. The last condition requires that the costs are not too large.

4) Robust attitude: As we observed from the previous example, game $\mathcal{A}$ may not have a dominant attitude for player i. In such a case, player i may prefer a strategy that guarantees the largest minimum ex-post reward. That is, player i might seek the *robust attitude* $\pi_i^\sharp \in [0, 1]$ defined by

$$\pi_i^\sharp := \arg\max_{\pi_i} \min_{\pi_j} u_i(x_i(\theta_i, \pi), x_j(\theta_j, \pi), \theta_i).$$

Theorem 4 *The robust attitude of Cournot duopoly does not coincide with pessimism and is given by*

$$\pi_i^\sharp = \min(1, (2 - 3\theta_i - \beta_i + 2\alpha_j)/4\Delta_j)$$

for $\Delta_j > 0$. Consequently, $\pi_i^\sharp > 0$, except for a singular case $\alpha_j = 0$ and $\theta_i = \beta_i = 1/2$.

Example 1 *Let $\beta := \max(\beta_i, \beta_j)$. Then if $\beta \le 1/4$, $\pi_i^\sharp = \pi_j^\sharp = 1$. That is, when costs are sufficiently small, the robust strategy is optimism. To see this, note that $\pi_i^\sharp = \min(1, (2 - 3\theta_i - \beta_i + 2\alpha_j)/4(\beta_j - \alpha_j)) \ge \min(1, (2 - 4\beta)/4\beta)) = 1$.*

IV. Existence of Uncertainty Equilibrium and its Relation to Nash Equilibrium

This section provides a condition for the existence of an uncertainty equilibrium.

Theorem 5 (Existence of Uncertainty Equilibrium) *Assume $r_i(X_j, \theta_i, \pi_i)$ is single-valued and continuous in X_j, θ_i and π_i. Then there exists an uncertainty equilibrium $(X_1^*(\pi), X_2^*(\pi))$.*

At an uncertainty equilibrium $(X_1^*(\pi), X_2^*(\pi))$, i's best response is

$$x_i^*(\pi) = r_i(X_j^*(\pi), \theta_i, \pi_i).$$

From the proof of Theorem 5, note there is one-to-one correspondence between $x_i^*(\pi)$'s and $X_i^*(\pi)$'s via r_i's. In particular, if Θ_i is a singleton, then $X_i^*(\pi) = x_i^*(\pi)$. This observation is stated in the next theorem.

Theorem 6 *Under the assumptions of Theorem 5, $\mathcal{G}(\pi)$'s uncertainty equilibrium $(X_1^*(\pi), X_2^*(\pi))$ coincides with game $\mathcal{G}_o$'s Nash equilibrium $(x_1^\star, x_2^\star)$ if $\Theta_i = \{\theta_i\}$ for $i = 1, 2$, irrespective of π.*

V. At least one player does not prefer pessimism

We identify conditions when pessimism cannot be dominant for both players.

The first theorem proves this for the non-symmetric Cournot duopoly game. The following theorem is for a more general utility structure of symmetric games.

Theorem 7 *Both Cournot duopoly players cannot simultaneously have pessimism as their dominant attitude.*

Now we consider a more general utility function case.

Theorem 8 *Consider a symmetric game where u_i is strictly monotonic in x_j and $r_i(x_j, \theta_i)$ is single valued and strictly monotonic in x_j and θ_i. Then pessimism cannot be a dominant attitude for any of the two players.*

VI. Conclusions

This paper proposes a framework to analyze two-player games with non-probabilisitc information uncertainty. The formulation allows a rational player to choose an attitude against uncertainty characterized by a degree of optimism. Corresponding to a pair of attitudes, we define an uncertainty equilibrium as a pair of sets of strategies from which rational players would not depart unilaterally. Under some assumptions, this concept coincides with the traditional Nash equilibrium when there is no uncertainty. We then define a two-phase game where players first choose their attitude. Finally, we illustrate the framework with an investment game and a Cournot duopoly game with uncertainty. We show that the framework may identify uniquely the strategies of the players.

Appendix

A. Proof of Theorem 1

The partial derivative with respect to x_i is $1 - \exp\{-\theta_i + x_i + x_j\}$, which is positive for $x_i < \theta_i - x_j$ and negative for $x_i > \theta_i - x_j$. Accordingly, the best response $x_i(x_j)$ is $x_i(x_j) = [\theta_i - x_j]^+$. If $\theta_i < \theta_j$, the only Nash equilibrium is then $x_i = 0, x_j = \theta_j$. The outcome of the game is very sensitive to the order of the parameters.

Assume i knows that $x_j \in X_j$. For $z \in \mathbb{R}$, define $[z]_0^1 := \min\{\max\{z, 0\}, 1\}$. Then, if i is optimistic, she maximizes $x_i - \exp\{-\theta_i + x_i + \alpha_j\}$ where $\alpha_j = \min X_j$. Thus,

$$x_i = [\theta_i - \alpha_j]_0^1 \in [[\alpha - \alpha_j]_0^1, [\beta - \alpha_j]_0^1].$$

Also, if i is pessimistic, she maximizes $x_i - \exp\{-\theta_i + x_i + \beta_j\}$ where $\beta_j = \max X_j$. Thus,

$$x_i = [\theta_i - \beta_j]_0^1 \in [[\alpha - \beta_j]_0^1, [\beta - \beta_j]_0^1].$$

Suppose both players are optimistic. Then the only uncertainty equilibrium is $X_i = X_j = [a, b]$ where $a =$

$\alpha - a$ and $b = \beta - \alpha$. Hence $X_i = X_j = [\frac{\alpha}{2}, \beta - \frac{\alpha}{2}]$. Consequently, $x_i = \theta_i - \frac{\alpha}{2}$ and

$$U_i(1,1) := \theta_i - \frac{\alpha}{2} - \exp\{\theta_j - \alpha\}.$$

Second, suppose both players are pessimistic. Then the only consistent sets are $X_i = X_j = [a, b]$ where $a = \alpha - b$ and $b = \beta - b$. Hence, $X_i = X_j = [\alpha - \frac{\beta}{2}, \frac{\beta}{2}]$. Consequently, $x_i = \theta_i - \frac{\beta}{2}$ and

$$U_i(0,0) := \theta_i - \frac{\beta}{2} - \exp\{\theta_j - \beta\}.$$

Third, suppose that player 1 is optimistic and player 2 is pessimistic. In that case, the only consistent sets are $X_1 = [a_1, b_1]$ and $X_2 = [a_2, b_2]$ where $a_1 = [\alpha - a_2]_0^1, b_1 = [\beta - a_2]_0^1, a_2 = [\alpha - b_1]_0^1, b_2 = [\beta - b_1]_0^1$. Hence, $X_1 = [\alpha, \beta]$ and $X_2 = \{0\}$. Consequently, $x_1 = \theta_1$ and $x_2 = \theta_2 - \beta$, so that

$$U_1(1,0) := \theta_1 - \exp\{\theta_2 - \beta\}.$$

By symmetry,

$$U_1(0,1) := \theta_1 - \beta - \exp\{\theta_2 - \beta\}.$$

By inspection, we see

$$U_1(1,0) \geq U_1(0,0) \text{ and } U_1(1,1) > U_1(0,1).$$

Thus, optimism is a dominant strategy for player 1. By symmetry, it is also dominant for player 2.

B. Proof of Theorem 2

The proof goes in following steps: First we define the uncertainty set as a ball. Then we show the ball's radius is constant. Finally we show the center of the ball is fixed at equilibrium. Note u_i is negatively affine in x_j. Let $X_o = [0, 1/2]$ be the strategy space. Thus $\inf X_j = \arg\sup_{x_j \in X_j} u_i(x, \theta_i)$ and $\sup X_j = \arg\inf_{x_j \in X_j} u_i(x, \theta_i)$. Define

$$h_i(X_j, \pi_i) = \pi_i \inf X_j + (1 - \pi_i) \sup X_j.$$

Then $f_i(x_i, X_j, \theta_i, \pi_i) = u_i(x_i, h_i(X_j, \pi_i), \theta_i)$. From the first order condition and definition, i's best response to X_j becomes

$$r_i(X_j, \theta_i, \pi_i) = (1 - h_i(X_j, \pi_i) - \theta_i)/2.$$

This yields

$$\sup X_i = (1 - r_i(X_j, \pi_i) - \alpha_i)/2$$
$$\inf X_i = (1 - r_i(X_j, \pi_i) - \beta_i)/2.$$

Now let $X_i^* = \mathcal{B}[s_i, t_i]$ for $i = 1, 2$ and $j \neq i$ where $\mathcal{B}[s, t]$ is a closed ball or radius t centered at s. Then

$$t_i = (\sup X_i^* - \inf X_i^*)/2 = \Delta_i/4,$$

where $\Delta_i := \beta_i - \alpha_i$. This is independent of $X_i^*, X_j^*, \theta_i, \theta_j$. Now since $\sup X_j^* = s_j + t_j$ and $\inf X_j^* = s_j - t_j$,

$$h_i(X_j^*, \pi_i) = s_j + t_j(1 - 2\pi_i).$$

Define $\sigma_i := (\alpha_i + \beta_i)/4,$. Then

$$\begin{aligned} s_i &= (\sup X_i^* + \inf X_i^*)/2 \\ &= (1 - r_i(X_j^*))/2 - \sigma_i = (1 - s_j - t_j(1 - 2\pi_i))/2 - \sigma_i \end{aligned}$$

for $i = 1, 2$. We have two equations relating s_i and s_j. By solving algebra, we get (10). i's best response at uncertainty equilibrium $x_i = r_i(X_j^*, \theta_i, \pi_i)$ becomes (11). $\mathcal{B}[s_i, t_i]$ then is uniquely determined by given $(\pi, \theta_i, \Theta_i, \Theta_j)$. To show its existence, it is sufficient to show $\mathcal{B}[s_i, t_i] \subset X_o$. To see this, it is straightforward to verify $\min s_i + t_i \leq \sup X_o$ and $\max s_i - t_i \geq \inf X_o$ for all combinations of π, Θ_1, Θ_2.

C. Proof of Lemma 1

1) We need to find a condition, without loss of generality, such that (i) $u_1(OO) \geq u_1(PO)$ and (ii) $u_1(OP) \geq u_1(PP)$ for every $\theta_2 \in [\alpha_2, \beta_2]$. By algebra,

$$u_1(OO) - u_1(PO) \geq \Delta_2[\underline{\theta}_1 - \theta_1]/12,$$

which is non-negative for all $\theta_1 \leq \underline{\theta}_1 := \frac{1}{3}(2 - \beta_1 + 4\alpha_2 - 2\beta_2)$ and for all θ_2. (ii) is immediate because

$$u_1(OP) - u_1(PP) \geq u_1(OO) - u_1(PO).$$

2) Similar development yields $\theta_i \geq \overline{\theta}_i$ where $\overline{\theta}_i := \frac{1}{3}(2 - \alpha_i - 2\alpha_j + 4\alpha_i)$.

D. Proof of Theorem 3

1) We show at least one player always have an incentive to deviate from (PP). This part of the theorem is true even for non-symmetric Θ_1 and Θ_2. Define $u_i(\pi) := u_i(x_i^*(\pi), x_j^*(\pi), \theta_i)$ and $\Delta = \beta - \alpha$. Suppose player 1 does not have the incentive to deviate from (PP). That is, $u_1(PP) \geq u_1(OP)$. Then we prove by showing $u_2(OP) > u_2(PP)$. From the proof of Lemma 1, $u_1(PP) \geq u_1(OP)$ is equivalent to $3\theta_1 \geq 2 - 3\alpha - 2\beta + 6\theta_2$. Then, $u_2(PO) - u_2(PP) = \frac{\Delta_1}{36}(2 - 3\alpha - 2\beta + 6\theta_1 - 3\theta_2) \geq \frac{\Delta_1}{36}(6 - 9\alpha - 6\beta + 9\theta_2) > 0$. The last inequality comes from the boundary condition $0 \leq \alpha \leq \theta_i \leq \beta \leq 1/2$.
2) We show that a rival player's optimistic attitude is always detrimental: $36(u_1(PP) - u_1(PO)) = \Delta(6x_1(PP)) + \Delta(6 - 6\theta_1 - 6x_1(PP) - 6x_2(PP) - \Delta) > 0$. We can similarly show $36(u_1(OP) - u_1(OO)) > 0$. At (PP), suppose one player has incentive to change to O. That change hurts the ex post utility of the other player. This concludes (PP) is pareto efficient.
3) We need to show $u_i(PP) > u_i(OO)$. To see this, $36(u_i(PP) - u_i(OO)) = 12\Delta x_1(PP) - \Delta(6 - 6\theta_1 - 6x_1(PP) - 6x_2(PP) - 2\Delta) = \Delta(2 + 2\alpha + 2\beta - 3\theta_1 - 3\theta_2) \geq 0$.
4) If $\beta \leq \max(1/3, 2\alpha)$, then $\underline{\theta}_i \geq \beta \geq \theta_i$ for all i, and importantly, this fact becomes a common knowledge. From Lemma 1, O is the dominant strategy. Together with 1), 2) and 3)), this constitutes a Prisoner's Dilemma game.

E. Proof of Theorem 4

$u_i(q^*(\pi), \theta_i)$ is non-increasing in π_j for all possible combinations of parameters. Thus u_i is minimized at $\pi_j = 1$. u_i is convex in π_i. From the first order condition, the result is immediately obtained.

F. Proof of Theorem 5

Since r_i is continuous in θ_i, and Θ_i is a bounded and closed interval, X_i is a closed interval. Let $X_i = [\underline{x}_i, \overline{x}_i] \subset X_{i,o}$, $\underline{x}_i \le \overline{x}_i$. We define a map $\phi(\underline{x}_i, \overline{x}_i) = (\underline{x}_i', \overline{x}_i')$ such that

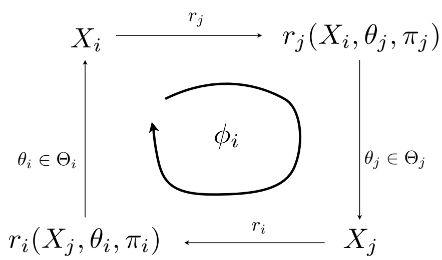

Fig. 1: ϕ_i mapping

$$\underline{x}_i' = \arg\min_{\theta_i \in \Theta_i} r_i([\underline{x}_j, \overline{x}_j], \theta_i, \pi_i)$$
$$\overline{x}_i' = \arg\max_{\theta_i \in \Theta_i} r_i([\underline{x}_j, \overline{x}_j], \theta_i, \pi_i)$$

where

$$\underline{x}_j = \arg\min_{\theta_j \in \Theta_j} r_j([\underline{x}_i, \overline{x}_i], \theta_j, \pi_j)$$
$$\overline{x}_j = \arg\max_{\theta_j \in \Theta_j} r_j([\underline{x}_i, \overline{x}_i], \theta_j, \pi_j).$$

From construction $\underline{x}_i' \le \overline{x}_i'$. If ϕ_i is a continuous mapping, then by Brouwer's fixed point theorem, there exists $(\underline{x}_i^*, \overline{x}_i^*) \in X_{i,o}^2$ such that

$$\phi_i(\underline{x}_i^*, \overline{x}_i^*) = (\underline{x}_i^*, \overline{x}_i^*).$$

Then $X_i = [\underline{x}_i^*, \overline{x}_i^*]$ is, by definition, an uncertainty equilibrium. Now we show that ϕ_i is continuous in $\underline{x}_i, \overline{x}_i$.

Let $v := y(\underline{x}_i, \overline{x}_i) := \arg\sup_{x_i \in [\underline{x}_i, \overline{x}_i]} u_j(x_j, x_i, \theta_j)$ and define z such that $\underline{x}_i - \epsilon \le z \le \underline{x}_i + \epsilon$. Then $\lim_{\epsilon \to 0} u_j(x_j, z, \theta_j) = u_j(x_j, \underline{x}_i, \theta_j)$ from u_j's continuity. There are two cases: (1) $y(\underline{x}_i, \overline{x}_i) > \underline{x}_i$. Then $y(z, \overline{x}_i) = y(\underline{x}_i, \overline{x}_i)$ as for small ϵ. (2) $y(\underline{x}_i, \overline{x}_i) = \underline{x}_i$. Then $\underline{x}_i - \epsilon \le w := y(z, \overline{x}_i) \le \underline{x}_i + \epsilon$. As a result

$$\sup_{x_i \in [z, \overline{x}_i]} u_j(x_j, x_i, \theta_j) - \sup_{x_i \in [\underline{x}_i, \overline{x}_i]} u_j(x_j, x_i, \theta_j)$$
$$= u_j(x_j, w, \theta_j) - u_j(x_j, x_i, \theta_j) \to 0$$

as $\epsilon \to 0$ from u_j's continuity.

Therefore $\sup_{x_i \in [\underline{x}_i, \overline{x}_i]} u_j(x_j, x_i, \theta_j)$ is continuous in $\underline{x}_i$. Similarly we can show it is continuous in $\overline{x}_i$. These steps can be repeated for $\inf_{x_i \in [\underline{x}_i, \overline{x}_i]} u_j(x_j, x_i, \theta_j)$. As a result f_j and r_j are continuous in $\underline{x}_i, \overline{x}_i$. Since r_j is continuous in θ_j and Θ_j is a closed and bounded interval, $X_j := [\underline{x}_j, \overline{x}_j] := \{r_j([\underline{x}_i, \overline{x}_i], \tilde{\theta}_j, \pi_j | \tilde{\theta}_j \in \Theta_j\}$ is a closed interval too. Using the same procedure, $\underline{x}_i'$ and $\overline{x}_i'$ are continuous in $\underline{x}_j, \overline{x}_j$. Since ϕ_i is a composite function of continuous functions in $\underline{x}_i, \overline{x}_i$, ϕ_i is therefore continuous in $(\underline{x}_i, \overline{x}_i)$. This completes the proof.

G. Proof of Theorem 6

Let $\Theta_i = \{\theta_i\}$ for all i. Then for arbitrary X_j, $X_i := \{r_i(X_j, \theta_i, \pi_i) | \theta_i \in \Theta_i\}$ is a singleton. Let $X_i = \{x_i^\dagger\}$. Then

$$x_j^\dagger := r_j(X_i, \theta_j, \pi_j) = \arg\max_{x_j \in X_{j,o}} u_j(x_j, x_i^\dagger, \theta_j)$$

is j's best response function of game $\mathcal{G}_o$ when j predicts i plays $x_i^\dagger$. By assumption an equilibrium of this is a $(x_1^\star, x_2^\star)$. And by construction, it is also an uncertainty equilibrium $(X_1^*(\pi), X_2^*(\pi))$ of $\mathcal{G}(\pi)$, and it does not depend on π.

H. Proof of Theorem 8

Suppose player 1's dominant attitude is pessimism. From Lemma 1, this implies

$$\beta_1 \ge \theta_1 \ge \overline{\theta}_1 = (2 - \alpha_1 + 4\beta_2 - 2\alpha_2)/3.$$

Now then,

$$\overline{\theta}_2 = (2 - \alpha_2 + 4\beta_1 - 2\alpha_1)/3$$
$$\ge (14 - 10\alpha_1 - 11\alpha_2 + 7\beta_2)/9 + \beta_2 > \beta_2.$$

Thus $\theta_2 \le \beta_2 < \overline{\theta}_2$. Therefore pessimism cannot be player 2's dominant strategy.

I. Proof of Theorem 8

Consider player 1 representatively. We will show $U_1(OO) > U_1(PO)$ for some $\theta_2 \in \Theta_2$. Let $u := u_i$, $r := r_i$ and $\Theta := [\alpha, \beta] = \Theta_i$ for $i = 1, 2$. $\alpha < \beta$. As one case, assume u_i is strictly decreasing in x_j, r_i is decreasing in x_j and θ_i both. The conclusion is the same if any of 'decreasing' condition is changed to 'increasing' condition. Define equilibrium sets for each π as follows:

$$X_1 = X_2 = [a, b] \text{ for } \pi = (OO)$$
$$X_1 = X_2 = [c, d] \text{ for } \pi = (PP)$$
$$X_1 = [e, f], X_2 = [g, h] \text{ for } \pi = (OP)$$
$$X_1 = [g, h], X_2 = [e, f] \text{ for } \pi = (PO).$$

Then

$$a = r(a, \beta) \text{ and } b = r(a, \alpha)$$
$$c = r(d, \beta) \text{ and } d = r(d, \alpha)$$
$$e = r(g, \beta) \text{ and } f = r(g, \alpha)$$
$$g = r(f, \beta) \text{ and } h = r(f, \alpha).$$

From monotonicity of r, we draw relation one by one: From $a = r(a, \beta)$ and $d = r(d, \beta)$, it is immediate to see $a < d$. Noting $d = r(r(d, \alpha), \alpha)$ and $g = r(r(g, \alpha), \beta)$, we get $g < d$. Thus $d < f$ from $d = r(d, \alpha)$ and $f = r(g, \alpha)$. From $a < f$, we get $g < a$. Finally we get $a < e$. Take $\theta_2 = \beta$. Then,

$$U_1(OO) = u(x_1(\theta_1, OO), x_2(\theta_2, OO), \theta_1)$$
$$= u(r(a, \theta_1), r(a, \theta_2), \theta_1)$$
$$= u(r(a, \theta_1), r(a, \beta), \theta_1)$$
$$= u(r(a, \theta_1), a, \theta_1)$$
$$> u(r(f, \theta_1), a, \theta_1)$$
$$> u(r(f, \theta_1), e, \theta_1) = U_1(PO)$$

Therefore pessimism cannot be a dominant attitude in a symmetric game.

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